

 MICADO	<p style="text-align: center;">MICADO Technical Note</p> <p style="text-align: center;">Atmospheric refraction and dispersion models for the MICADO ADC</p>	<table><tr><td>Doc.-Ref.</td><td>: ELT-PLA-MCD-56305-0012</td></tr><tr><td>Issue</td><td>: 1.0</td></tr><tr><td>Date</td><td>: 27.06.2019</td></tr><tr><td>Author</td><td>: J.A. van den Born</td></tr><tr><td>Page</td><td>: 1 of 12</td></tr></table>	Doc.-Ref.	: ELT-PLA-MCD-56305-0012	Issue	: 1.0	Date	: 27.06.2019	Author	: J.A. van den Born	Page	: 1 of 12
Doc.-Ref.	: ELT-PLA-MCD-56305-0012											
Issue	: 1.0											
Date	: 27.06.2019											
Author	: J.A. van den Born											
Page	: 1 of 12											

CHANGE RECORD

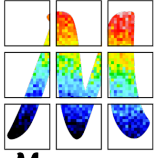
ISSUE	DATE	SECTION/PARAGRAPH AFFECTED	REASON/INITIATION DOCUMENTS/REMARKS
1.0	16.05.2018	All	New document
1.1	27.06.2018	5.5, 6.1, 6.2, 7	Reworked full integration method to improve accuracy; updated figures and tables accordingly; updated conclusion; added link to source code

1 Scope

The size of the ELT primary mirror and the wavelengths of interest for the MICADO instrument make the ADC an essential component for imaging observations. It is well known that the atmospheric model implemented in Zemax OpticStudio is reasonably valid for optical design, but is severely outdated when it comes to high accuracy, as its calculations can be traced back to the Barrel and Sears paper from 1939 [RD1]. For a relative comparison of various dispersion models, see for example Spanò [RD2].

Most instruments that currently employ an ADC only need to put it in roughly the right position to achieve good results, but the high required performance of MICADO a detailed study of dispersion effects is necessary to determine how carefully the dispersion has to be determined.

This document gives an overview of the different refraction and dispersion models that have been used in literature and investigates if they can be used for the MICADO ADC. The details of each model are not discussed in this document, but they can be requested from the author.

	<p style="text-align: center;">MICADO Technical Note</p> <p style="text-align: center;">Atmospheric refraction and dispersion models for the MICADO ADC</p>	Doc.-Ref. Issue Date Author Page	: ELT-PLA-MCD-56305-0012 : 1.0 : 27.06.2019 : J.A. van den Born : 2 of 12
---	---	--	---

2 Reference Documents

Sources of Uncertainty in Atmospheric Dispersion, a private document by the author, can be requested for more mathematical detail of the discussed models.

In order of appearance in the text:

Ref. no.	Description of reference document
[RD1]	Barrell, H. and Sears, J. E. (1939). <i>The Refraction and Dispersion of Air for the Visible Spectrum</i> . Philosophical Transactions of the Royal Society of London Series A, 238:1–64.
[RD2]	Spanò, P. (2014). <i>Accurate astronomical atmospheric dispersion models in ZEMAX</i> . In Advances in Optical and Mechanical Technologies for Telescopes and Instrumentation, volume 9151 of Proc. of SPIE, page 915157.
[RD3]	Ciddor, P. E. (1996). <i>Refractive index of air: new equations for the visible and near infrared</i> . Applied Optics, 35:1566.
[RD4]	Birch, K. P. and Downs, M. J. (1993). <i>An Updated Edlén Equation for the Refractive Index of Air</i> . Metrologia, 30:155–162.
[RD5]	Davis, R. S. (1992). <i>Equation for the Determination of the Density of Moist Air (1981/91)</i> . Metrologia, 29:67–70.
[RD6]	Corbard, T., Ikhlef, R., Morand, F., Meftah, M., and Renaud, C. (2019). <i>On the importance of astronomical refraction for modern solar astrometric measurements</i> . MNRAS, 483:3865–3877.
[RD7]	Auer, L. H. and Standish, E. M. (2000). <i>Astronomical Refraction: Computational Method for All Zenith Angles</i> . Astronomical Journal, 119(5):2472–2474.
[RD8]	NOAA (1976). <i>U.S. standard atmosphere, 1976</i> . Technical report.
[RD9]	Sandrock, S., Amestica, R., and Sarazin, M. (1999). VLT astronomical site monitor asm data user manual. ESO. Doc. No.: VLT-MAN-ESO-17440-1773.

3 Definitions

Atmospheric dispersion means the difference in refraction between two rays with different wavelength.

Atmospheric refraction means the change of direction of an incoming ray, due to refraction in the atmosphere. This is dependent on the atmospheric conditions and the direction of observation (zenith angle).

Dispersion means the change in the index of refraction of a medium due to variations in any of the relevant parameters, typically wavelength.

Refractivity is defined as the index of refraction minus unity, $R = n - 1$.

4 Overview of the Ciddor dispersion model

The model on refractivity of the air by Ciddor [RD3] is adopted by the International Association of Geodesy (IAG) as the standard set of equations to calculate the refractivity of the atmosphere. This model is generally considered applicable over a wider range of conditions than the more often used Birch & Downs model [RD4] from a few years prior.

The basics steps of this set of equations are as follows:

1. Calculate the refractivity for standard dry air ($T = 288.15$ K, $P = 101\,325$ Pa, $H = 0\%$ and CO_2 density is 450 ppm)
2. Calculate the corresponding refractivity for standard air, but with the measured CO_2 density.
3. Calculate the refractivity of water vapor at standard conditions ($T = 293.15$ K, $P = 1333$ Pa).
4. Measure or calculate the density of the air, taking into account the temperature, pressure, relative humidity and CO_2 density.
5. Use the Gladstone-Dale relation or the Lorentz-Lorentz relation to correlate the density of the air with the index of refraction of that same air.
6. Then a ratio between the densities for standard conditions and the actual conditions for both the dry air component, as well as the water vapor component allow the index of refraction to be determined.

Ciddor used the CIPM-81/91 density equation [RD5] to perform step 4. Since then an updated version has been adopted, taking also into account the Argon content of the air. These additions are small and likely do not affect the accuracy of the Ciddor model much, though in principle it is easy to modify the model to use these equations.

Table 1 shows three results calculated with the Ciddor model at $1.35\ \mu\text{m}$.

Table 1 Some results for the refractivity calculated with the Ciddor model. The first line shows the refractivity for standard dry air, the other two lines show some conditions that could be encountered on Cerro Armazones.

P (Pa)	T (K)	H	$(n - 1) \times 10^8$
101 325	288.15	0.00	27 346.5
71 200	279.65	0.22	19 790.4
71 200	279.65	0.60	19 775.8

 MICADO	<div><div>MICADO Technical Note</div><div>Atmospheric refraction and dispersion models for the MICADO ADC</div></div>	<div><div>Doc.-Ref.</div><div>Issue</div><div>Date</div><div>Author</div><div>Page</div></div> <div><div>: ELT-PLA-MCD-56305-0012</div><div>: 1.0</div><div>: 27.06.2019</div><div>: J.A. van den Born</div><div>: 4 of 12</div></div>
---	---	--

5 Atmospheric refraction models

5.1 Plane-parallel atmosphere

The simplest method to calculate atmospheric refraction is by assuming that the atmosphere behaves like a plane parallel surface. Sir Isaac Newton first discovered that the different indices of refraction of multiple sequential media do not influence the final refraction. Only the initial index of refraction, which in this case will be unity due to the vacuum of space, and the final index of refraction, which will be the observed index of refraction as measured or calculated with the Ciddor model, are involved.

The plane-parallel model is only valid for small zenith angles. The error grows larger as the zenith distance increases. The refraction r in this model is easily derived from Snell's law.

$$r = z_H - z_o = \sin^{-1}(n \sin z_o) - z_o$$

Where the refraction is the difference between the observed zenith angle, z_o and the local zenith angle, before the ray gets refracted, z_H .

5.2 Cassini's homogeneous atmosphere

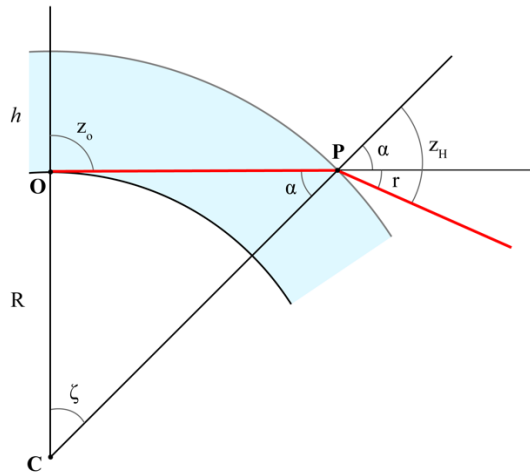


Figure 1 Diagram for a homogeneous spherical atmosphere. The radius of curvature of the earth is given by R , the height of the atmosphere is h and the incoming ray has a local zenith distance z_H , before it gets refracted and observed at a zenith distance z_o .

In reality a better approximation is made by assuming the atmosphere is spherical. Using the geometry in Figure 1 and the Sine law we find the formulation for Cassini's homogeneous atmosphere refraction model.

$$r = z_H - \alpha = \sin^{-1}\left(\frac{nR \sin z_o}{R + h}\right) - \sin^{-1}\left(\frac{R \sin z_o}{R + h}\right)$$

Assumed here is that the atmosphere is made out of a single homogeneous layer. In practice this is not as big an assumption as one might initially believe. Under the more reasonable assumption of ideal gas behavior and by invoking hydrostatic equilibrium we can calculate the height corresponding to such a homogeneous atmosphere for the conditions as the observatory. This is called the *reduced height*.

$$h = \frac{p_o}{\rho_o g},$$

with g the local gravitational acceleration. Typically, h is about 8 km.

 MICADO	<p>MICADO Technical Note</p> <p>Atmospheric refraction and dispersion models for the MICADO ADC</p>	<p>Doc.-Ref. : ELT-PLA-MCD-56305-0012</p> <p>Issue : 1.0</p> <p>Date : 27.06.2019</p> <p>Author : J.A. van den Born</p> <p>Page : 5 of 12</p>
---	---	---

Alternatively, one could calculate the height corresponding to an isothermal atmosphere. In that case the isothermal scale height is used.

$$h = \frac{k_b T_0}{mg},$$

where k_b is the Boltzmann constant, T_0 is the temperature at the observatory and m is the mass in kilograms of an average molecule of air.

5.3 The $\tan^3$ and $\tan^5$ refraction models

The most common form of the atmospheric refraction is of the following form

$$r = A \tan(z) + B \tan^3(z),$$

where A and B are constants. These constants can be retrieved by observations of at least two sources or they can be found analytically from the refraction integral and the Gladstone-Dale relation. For higher precision an additional third term can be considered as well, though this is typically not done as it does not give much additional accuracy for typical zenith angles at which telescopes observe. This leads to

$$r = \alpha(1 - \beta) \tan(z) - \alpha \left(\beta - \frac{\alpha}{2} \right) \tan^3(z) + 3\alpha \left(\beta - \frac{\alpha}{2} \right)^2 \tan^5(z),$$

With $\alpha = n - 1$ and $\beta = h/R$.

5.4 Error function refraction model

In Corbard et al. [RD6] a model is described that does not converge rapidly for large zenith angles and which involves the *Gauss error function*. It assumes an exponential decrease in the density of the atmosphere with height. The refraction equation described here can retrieve the models from the previous section quite easily.

$$r = \alpha \left(\frac{2 - \alpha}{\sqrt{2\beta - \alpha}} \right) \sin(z) \Psi \left(\frac{\cos(z)}{\sqrt{2\beta - \alpha}} \right)$$

Where

$$\Psi(x) = e^{x^2} \int_x^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} e^{x^2} (1 - \operatorname{erf}(x))$$

and as before we have $\alpha = n - 1$ and $\beta = h/R$. Here we used the Gauss error function $\operatorname{erf}(x)$ for notational simplicity.

 MICADO	<div>MICADO Technical Note</div> <div>Atmospheric refraction and dispersion models for the MICADO ADC</div>	<div>Doc.-Ref.</div> <div>Issue</div> <div>Date</div> <div>Author</div> <div>Page</div> <div>: ELT-PLA-MCD-56305-0012</div> <div>: 1.0</div> <div>: 27.06.2019</div> <div>: J.A. van den Born</div> <div>: 6 of 12</div>
---	---	--

5.5 Refraction integral

Without making assumptions about the structure of the atmosphere it can be easily shown from geometric arguments that the total refraction experienced by a light ray can be calculated by

$$r = \int_1^{n_o} \tan(z) \frac{dn}{n}.$$

Here the zenith angle z is the local zenith angle at any height, which changes as the light travels through the atmosphere. Rewriting the $\tan(z)$ term and making use of the properties of the refractive invariant $nR \sin(z) = n_o R_o \sin(z_o)$ we may write the equation above as

$$r = \int_1^{n_o} \frac{n_o R_o \sin(z_o) dn}{n \sqrt{(nR)^2 - (n_o R_o \sin(z_o))^2}}.$$

Auer & Standish [RD7] have shown that the calculation can be made computationally easier and more stable by a changing the variable of integration from the refractive index to the zenith angle and evaluating the integral in log-space. This then results in

$$r = - \int_0^{z_o} \frac{(d \ln n)/(d \ln r)}{1 + (d \ln n)/(d \ln r)} dz$$

If we want to use this model, we must know the conditions at any height in the atmosphere. A good starting point here is to use the 1976 U.S. Standard Atmosphere [RD8]. Unfortunately, the Ciddor model is not good enough to calculate the index of refraction for any T , P , H or ρ (density). We could instead use the Gladstone-Dale relation again

$$n = 1 + A\rho$$

Where we calculate the value of the constant A using the Ciddor model for the conditions on the ground.

 MICADO	<p style="text-align: center;">MICADO Technical Note</p> <p style="text-align: center;">Atmospheric refraction and dispersion models for the MICADO ADC</p>	<table><tr><td>Doc.-Ref.</td><td>: ELT-PLA-MCD-56305-0012</td></tr><tr><td>Issue</td><td>: 1.0</td></tr><tr><td>Date</td><td>: 27.06.2019</td></tr><tr><td>Author</td><td>: J.A. van den Born</td></tr><tr><td>Page</td><td>: 7 of 12</td></tr></table>	Doc.-Ref.	: ELT-PLA-MCD-56305-0012	Issue	: 1.0	Date	: 27.06.2019	Author	: J.A. van den Born	Page	: 7 of 12
Doc.-Ref.	: ELT-PLA-MCD-56305-0012											
Issue	: 1.0											
Date	: 27.06.2019											
Author	: J.A. van den Born											
Page	: 7 of 12											

6 A comparison of the different refraction models

All mathematical equations discussed in the previous section have been translated to Python code. A verification that a given set of input parameters gave the same output as in the original papers was performed. From the refraction the dispersion can easily be calculated, as this is simple the difference in refraction between two different wavelengths. An overview of all the parameters used is given below in Table 2.

Table 2 Description of the different parameters used in the complete Python model.

Parameter	Description
T	Temperature at the observer in Kelvin.
P	Pressure at the observer in Pascals.
H	Relative humidity at the observer, between 0 and 1.
x_c	CO ₂ content in the air in parts per million.
Latitude	Latitude of the observer, used for a correction of the Earth's radius of curvature due to elongation of the Earth near the equator.
h_0	Height of the observatory.
λ_1	Blue end of the observing wavelength band, for which the atmospheric dispersion is to be calculated.
λ_2	Red end of the observing wavelength band, for which the atmospheric dispersion is to be calculated.
z	Zenith angle of the observation.

6.1 Refraction

For standard (dry) air we calculate the refraction using all available refraction models. The dispersion is calculated using the Ciddor model, as far as this is possible. For the full integration method we use the 1976 U.S. Standard Atmosphere. Table 3 shows the refraction of the different models for various zenith angles. Figure 2 shows the refraction at very large zenith angles. Here it is clearly illustrated that the different models quickly diverge at these extreme angles.

Table 3 The refraction at $\lambda=1.49 \mu\text{m}$ for the different refraction models at standard (dry) air conditions. Except for the zenith angles all values are in arc seconds.

z	Plane-parallel	Cassini	tan3	tan5	Error function	Full integration
10	9.940	9.926	9.926	9.926	9.926	9.926
30	32.547	32.489	32.489	32.489	32.490	32.489
50	67.193	66.977	66.977	66.977	66.978	66.978
60	97.677	97.160	97.157	97.161	97.164	97.162
70	155.038	153.298	153.274	153.312	153.321	153.314
80	321.112	307.982	306.982	308.399	308.253	308.142
85	656.255	561.900	542.875	589.985	571.503	570.005

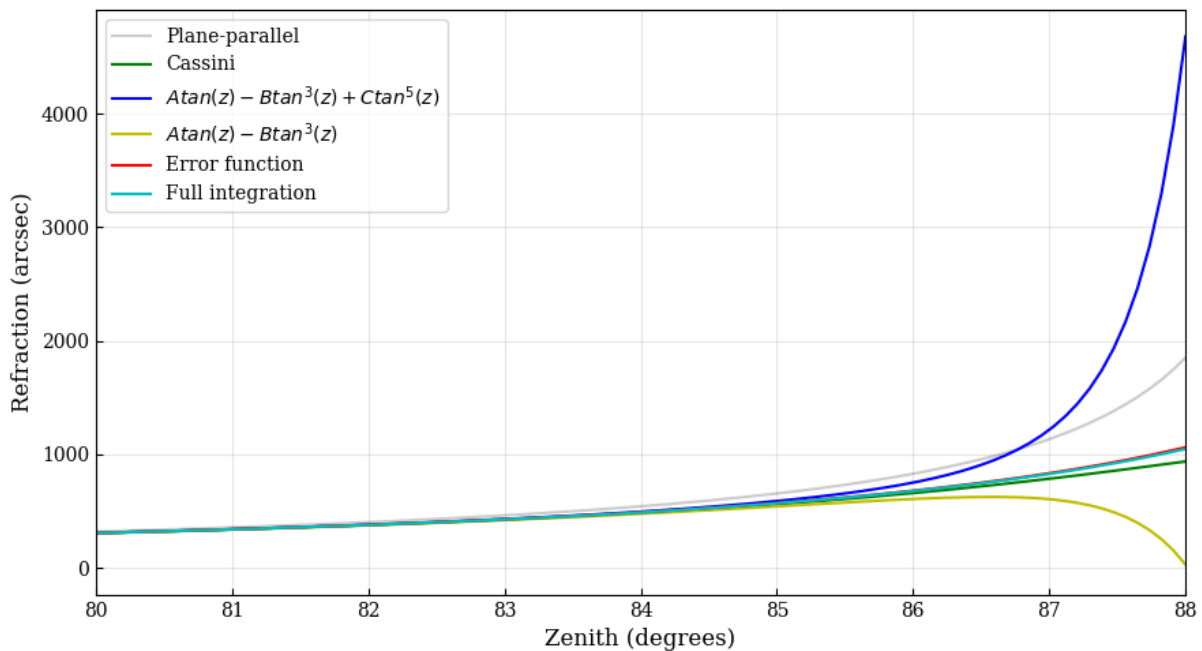


Figure 2 A comparison of the different refraction models for very large zenith angles. This is the domain where all models diverge rapidly. The refraction is calculated at $\lambda=1.49 \mu\text{m}$.

6.2 Dispersion

For the dispersion the actual level of refraction does not matter, since we're only looking at the differential refraction. This is why the differences between the different models are much smaller for dispersion.

Table 4 and Figure 3 show this for H-band and at various zenith angles, again at standard (dry) air conditions.

Table 4 The dispersion in H-band for the different refraction models at standard (dry) air conditions. Except for the zenith angles all values are in milli arc seconds.

z	Plane-parallel	Cassini	tan3	tan5	Error function	Full integration
10	7.565	7.554	7.554	7.554	7.554	7.554
30	24.771	24.727	24.727	24.727	24.727	24.727
50	51.147	50.982	50.982	50.982	50.983	50.983
60	74.367	73.973	73.971	73.973	73.975	73.974
70	118.112	116.784	116.770	116.792	116.797	116.794
80	245.468	235.120	234.695	235.526	235.459	235.421
85	508.856	433.512	421.898	449.557	439.625	439.090

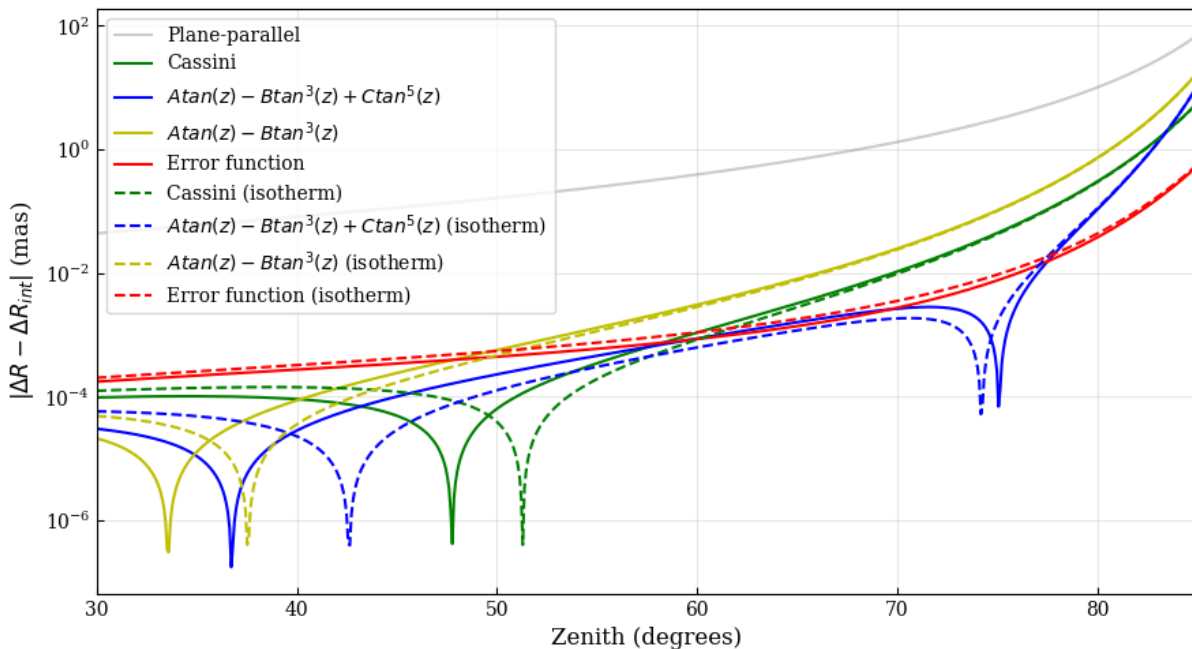


Figure 3 The dispersion in H-band relative to the full integration method as function of zenith angle for the different refraction models. For all zenith angles smaller than 60° the differences between the different methods is less than a few micro arcseconds, except for the plane-parallel approximation.

6.3 Impact of temperature and pressure variations at high altitudes

The data from the 1976 U.S. Standard Atmosphere contains, among others, temperature, pressure and density information at heights from slightly below sea level up to 86 km. The standard atmosphere considers eight different layers, which can most clearly be observed by the temperature inversions that take place.

The full integration method in combination with the data of the 1976 U.S. Standard Atmosphere allows us to investigate the effect of perturbations in temperature and pressure at higher altitudes. We do this by adding gaussian noise to both temperature and pressure at four altitudes, being 1-2 km, 7-10 km, 25-30 km and 50-60 km, respectively. The corresponding mean of the random gaussian noise is zero for each set. The standard deviation is set to 2, 3, 5 and 3 Kelvin for the temperature perturbations and to 1000, 500, 250 and 4 Pascals for the pressure perturbations respectively. See also Figure 4 and Figure 5. The used altitude resolution is 500 meters. The conditions at ground level are kept at the standard dry air conditions. Since the density of the atmosphere at higher altitudes is very low, the refractivity is also very low and thus a very large relative change is needed for a significant impact on the refraction and dispersion. Most of the refraction takes place between 0 and 10 km, where the atmospheric density is highest. For the perturbations above we find no significant change in the overall refraction, where the change compared to the non-perturbed atmosphere is less than 5 μ as, nor in the dispersion, for which the change is on the order of nano arc seconds. These results change slightly if the standard deviation of the added noise is altered, but the conclusions remain for any realistic perturbations. At these orders of magnitude the effects of atmospheric seeing, of which residuals will still exist after the AO system has corrected for most of these effects, will certainly be dominant. This shows that it won't be necessary to measure high altitude weather, nor is it necessary to perform a full integration along the atmosphere to calculate the dispersion.

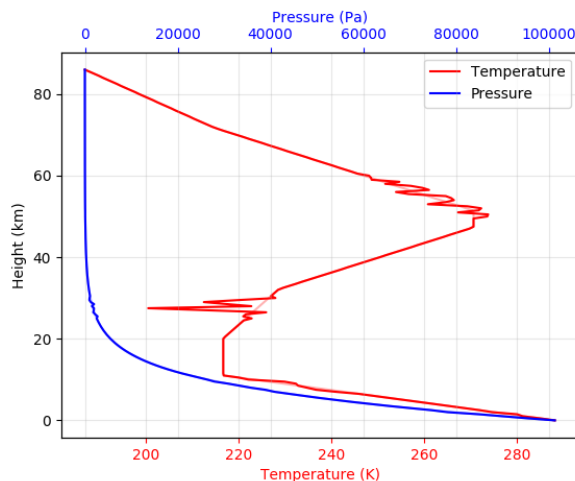


Figure 4 The temperature (red) and pressure (blue) as a function of altitude, as given by the 1976 U.S. Standard Atmosphere, but perturbed with gaussian noise at four zones. In lighter color the original data is shown.

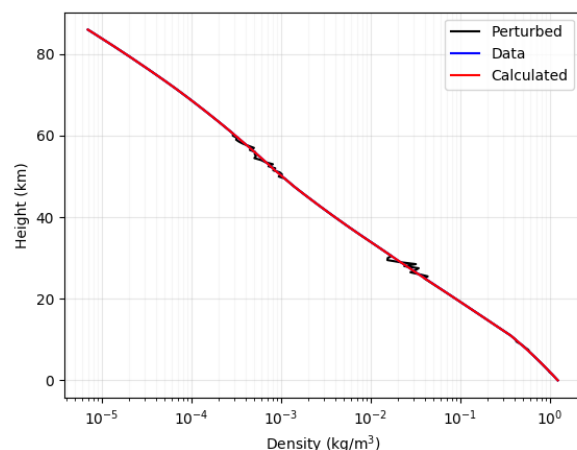
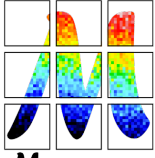


Figure 5 The density of air as given by the 1976 U.S. Standard Atmosphere. Shown are the perturbed data (black), the original data (blue) and the density as calculated from the temperature and pressure data (red).

	<p style="text-align: center;">MICADO Technical Note</p> <p style="text-align: center;">Atmospheric refraction and dispersion models for the MICADO ADC</p>	<p>Doc.-Ref. : ELT-PLA-MCD-56305-0012 Issue : 1.0 Date : 27.06.2019 Author : J.A. van den Born Page : 11 of 12</p>
---	---	--

6.4 Error propagation

For the Ciddor-Cassini refraction and dispersion model we have performed analytical error propagation, using the classic formula

$$\delta R(x_0, x_1, \dots) = \sqrt{\left(\frac{\partial R}{\partial x_0} \delta x_0\right)^2 + \left(\frac{\partial R}{\partial x_1} \delta x_1\right)^2 + \dots},$$

where $x_0, x_1, \dots$ are the variables of the function R and $\delta x_0, \delta x_1, \dots$ are the corresponding measurement uncertainties.

It was verified by means of a Monte Carlo simulation that the error propagation was performed properly and the results showed a difference of at most a few percent for various atmospheric conditions, both for the refractive index as well as the atmospheric refraction and atmospheric dispersion following the Cassini atmospheric model.

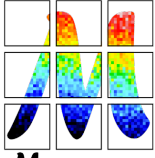
This allows us to investigate how accurately the conditions must be known in order to sufficiently correct the atmospheric dispersion, using the ADC. Uncertainties in T , P , H , λ_1 , λ_2 and the zenith angle are allowed. Table 5 shows the dispersion in H-band for representative conditions at Cerro Paranal for a range of measurement uncertainties. Most notable is the contribution of the uncertainty in the edges of the bandpass to the overall dispersion uncertainty.

In addition, we investigated the uncertainty in the zenith angle. Any zenith uncertainty less than an arcminute adds only a few μas of dispersion uncertainty to the overall dispersion uncertainty. For the VLT the pointing accuracy of the telescope is 3 arcseconds RMS, so if we assume a similar value, then the knowledge about the pointing position of the ELT should not be a problem.

We can conclude that these realistic uncertainties do not seem to pose a problem for the imaging requirements of MICADO. For the astrometric requirements, these uncertainties should be included in the overall astrometric budget.

Table 5 The dispersion in H-band at a zenith distance of 30° , as calculated using the Ciddor dispersion model and the Cassini atmospheric model, including uncertainties in the measured observables. Uncertainties not mentioned are $x_c=450\pm 20$ ppm and $\delta\lambda=1$ nm. No uncertainty in the zenith distance is assumed. The first line takes no uncertainties, the fourth line takes the uncertainties from the VLT Site Monitor on Paranal [RD9].

P	T	H	δP	δT	δH	ΔR	$\delta(\Delta R)$	$\delta(\Delta R) (\delta\lambda=0)$
(Pa)	(K)		(Pa)	(K)		(mas)	(μas)	(μas)
71200	279,65	0,22	0	0	0	17,922	0	0
71200	279,65	0,22	20	0,2	0,02	17,922	94,3	14,2
71200	279,65	0,22	10	0,1	0,01	17,922	93,5	7,1
71200	279,65	0,22	10	0,2	0,01	17,922	94,2	13,5
71200	279,65	0,22	100	1	0,05	17,922	117	70,8

 MICADO	<p style="text-align: center;">MICADO Technical Note</p> <p style="text-align: center;">Atmospheric refraction and dispersion models for the MICADO ADC</p>	Doc.-Ref. Issue Date Author Page	: ELT-PLA-MCD-56305-0012 : 1.0 : 27.06.2019 : J.A. van den Born : 12 of 12
--	---	--	--

7 Conclusion

We have discussed the dispersion model as described by Ciddor in 1996 and applied this to various refraction models. Although the models diverge quickly at large zenith angles, they agree very well for the elevation angles that a typical telescope will observe at. Even for the large next generation of telescopes this will likely not pose a problem. We have also concluded that monitoring the weather at high altitudes in the atmosphere will not be necessary from the standpoint of dispersion control. Finally, the contribution of measurement uncertainties on the overall dispersion uncertainty was investigated. It was shown that the uncertainty in wavelength is the biggest contributor to the dispersion error. For imaging this should pose no problem to the requirements, though these findings should likely be included in the astrometry budget.

The full integration method to calculate the refraction and dispersion requires complete knowledge of the atmosphere at any height and is the most computationally expensive. For this analysis this method was only used to illustrate the negligible effect of pressure and temperature variation at high altitudes. Historically used methods to calculate the dispersion are still valid, though a personal preference for the Cassini model exists, due to its mathematical simplicity.

The mathematical framework discussed in this document and the corresponding Python code are available from the author or can be downloaded at <https://gitlab.astro-wise.org/micado/atmosphericmodel>

End of Document